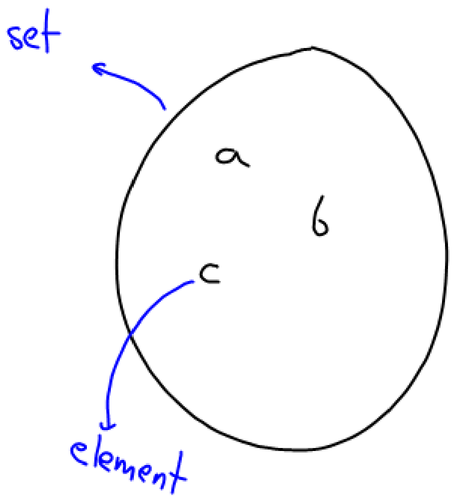
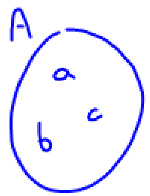
Sets



Relation $\equiv$ law or simple relation between two elements of a set.

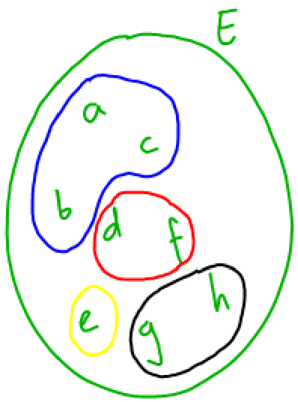


$a R b$

Equivalence relations $\equiv$ A relation that follows:

1. Reflection $\quad aRa \quad \forall a \in E$
2. Symmetry $\quad aRb \longrightarrow bRa \quad \forall a,b \in E$
3. Transitivity $\quad aRb \wedge bRc \longrightarrow aRc \quad \forall a,b,c \in E$

Example:



Given:

aRb

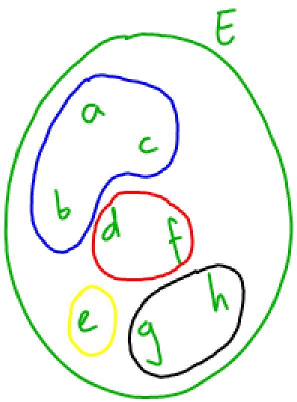
cRa

dRf

gRh

If we know R is an ER, what does R do with E.

bRc



R forms a Partition in E : E/R

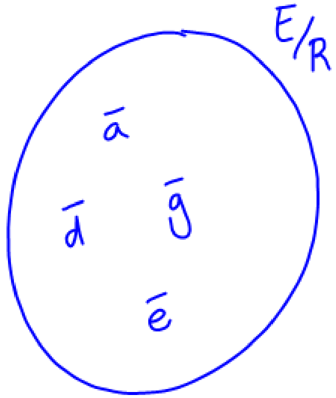
<u>Equivalence class</u> : $\bar{x} = \{ y \in E \,/\, x R y \}$

$\bar{a} = \{ a, b, c \} = \bar{b} = \bar{c}$

$\bar{d} = \{ d, f \} = \bar{f}$

$\bar{g} = \{ g, h \} = \bar{h}$

$\bar{e} = \{ e \}$



Groups

A group is a set with an ILC (internal law of composition)

$(\underset{\text{set}}{G}, \underset{\text{ILC}}{*})$

$*$ is an ILC $\Longleftrightarrow$
- Internal $\forall a,b \in G \quad a*b \in G$
- $\exists$ Neutral element $\exists!\, e \in G \,/\, \forall a \in G \quad a*e = e*a = a$
- Symmetrical $\forall a \in G \quad \exists a' \in G \,/\, a*a' = a'*a = e$

Example: $(\mathbb{Z}, +)$

Example:

$G = \{1, -1, i, -i\}$

$\cdot \equiv$ Natural multiplication of $\mathbb{C}$

$\cdot$	1	-1	i	-i
1	1	-1	i	-i
-1	-1	1	-i	i
i	i	-i	-1	1
-i	-i	i	1	-1

INTERNAL ✓

Neutral: 1 ✓

1 Autosymmet.
-1 Autosymmet.
i, -i Symmet. Pair
✓

Autosymmetrical ≡ an element that acts as its own symmet. $\forall a \in G \quad a = a' \quad a * a = e$

Abelian Group ≡ group where $\forall a, b \in G \quad a * b = b * a$

<u>Finite groups</u> Groups that have a finite number of elements called the group's ORDINAL (or order).

$(G = \{1, -1, i, -i\}, \cdot) \qquad o(G) = 4$

<u>Order of an element</u> Number of times we must write an element operated with itself to obtain the neutral element.

$(G, \cdot) \qquad o(i) = 4$ because $i \cdot i \cdot i \cdot i = 1$

<u>Lagrange's Law</u>. Inside a finite group the order of any element is a divisor of the order of the group.

$(G, \cdot)$

$o(G) = 4$

$o(1) = 1$

$o(-1) = 2$

$o(i) = 4$

$o(-i) = 4$

→ The Neutral element ALWAYS has order 1

→ Autosymmetrical elements ALWAYS have order 2

→ Symmetrical elements ALWAYS have the same order

Subgroups Given a group $(G, *)$, we can say that $S \subseteq G$ is a subgroup of $G \iff (S, *)$ is a Group.

contained or equal to

Also: $\boxed{S \text{ is subgroup of } G \iff \forall a, b \in S \quad a * b^{-1} \in S}$

Lagrange's law for subgroups: A subgroup's order is always divisor of the groups order.

Example: $(G, \cdot)$

$G = \{1, -1, i, -i\}$

Orders of subgroups can only be 1, 2 and 4

$o(S_1) = 1 \longrightarrow S_1 = \{1\}$

$\cdot$	1
1	1

$o(S_2) = 2 \longrightarrow S_2 = \{1, -1\}$

$\cdot$	1	-1
1	1	-1
-1	-1	1

$o(S_3) = 4 \longrightarrow S_3 = \{1, -1, i, -i\}$

Cyclical groups If a finite group has an element of the same order as the group itself, that group is called cyclical, and that element is called a generator.

$$(G, \cdot) \quad \left.\begin{matrix} o(G) = 4 \\ o(i) = 4 \end{matrix}\right\} \longrightarrow G \text{ is cyclical} \longrightarrow i \text{ is a generator of } G$$

i

$i \cdot i = -1$

$i \cdot i \cdot i = -i$ 1st

$i \cdot i \cdot i \cdot i = 1$

$i \cdot i \cdot i \cdot i \cdot i = i$

$i \cdot i \cdot i \cdot i \cdot i \cdot i = -1$

$i \cdot i \cdot i \cdot i \cdot i \cdot i \cdot i = -i$ 2nd

$i \cdot i \cdot i \cdot i \cdot i \cdot i \cdot i \cdot i = 1$

$i \cdot i \cdot i \cdot i \cdot i \cdot i \cdot i \cdot i \cdot i = i$

$\vdots$

Normal Subgroups When a subgroup $H \subset G$ follows these rules, it is called a Normal Subgroup:

$$\begin{cases} 1.\ \forall a \in G \quad a * H \subseteq H \\ \qquad \text{OR} \\ 2.\ \forall a \in G \quad H * a \subseteq H \\ \qquad \text{OR} \\ 3.\ \forall a \in G \quad a * H = H * a \end{cases}$$

If G is Abelian then every subgroup in G is NORMAL.